Programme 4

Laplace transforms 3

Frames 1 to 70

Learning outcomes

When you have completed this Programme you will be able to:

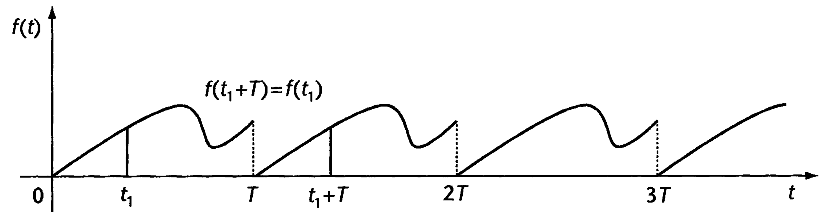
- Find the Laplace transforms of periodic functions
- Obtain the inverse Laplace transforms of transforms of periodic functions
- Describe and use the unit impulse to evaluate integrals
- Obtain the Laplace transform of the unit impulse
- Use the Laplace transform to solve differential equations involving the unit impulse
- Solve the equation and describe the behaviour of an harmonic oscillator

Laplace transforms of periodic functions

1

Periodic functions

Let $f(t)$ represent a periodic function with period T so that $f(t+nT) = f(t)$ with a graph of the following form



If we describe the first cycle by $\bar{f}(t)$ then

$$\bar{f}(t) = \begin{cases} f(t) & \text{for } 0 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

The second cycle is identical to the first cycle except that it is shifted by T units of time along the t-axis. Therefore the second cycle can be described in terms of the Heaviside unit step function as $\bar{f}(t-T)u(t-T)$. That is

$$\bar{f}(t-T)u(t-T) = \begin{cases} f(t) & \text{for } T \leq t < 2T \\ 0 & \text{otherwise} \end{cases}$$

By this reasoning the periodic function $f(t)$ is represented by

$$f(t) = \bar{f}(t)u(t) + \ldots\ldots\ldots\ldots$$

2

$$\boxed{f(t) = \bar{f}(t)u(t) + \bar{f}(t-T)u(t-T) + \bar{f}(t-2T)u(t-2T) + \cdots}$$

Because

$u(t)$ switches on $\bar{f}(t)$ at time $t=0$, $u(t-T)$ switches on $\bar{f}(t-T)$ at time $t=T$ and $u(t-2T)$ switches on $\bar{f}(t-2T)$ at time $t=2T$, etc.

Consider now the Laplace transform of $\bar{f}(t)$. By definition

$$L\{\bar{f}(t)\} = \int_0^\infty e^{-st}\bar{f}(t)\,\mathrm{d}t = \int_0^T e^{-st}f(t)\,\mathrm{d}t = \bar{F}(s)$$

because for $t > T$, $\bar{f}(t) = 0$ and so the semi-infinite integral becomes an integral just over the period of $f(t)$. Using the second shift theorem (see Frame 10 of Programme 3), the Laplace transform of $f(t)$ is

$$L\{f(t)\} = L\{\bar{f}(t)u(t)\} + L\{\bar{f}(t-T)u(t-T)\} + L\{\bar{f}(t-2T)u(t-2T)\} + \cdots$$

That is

$$L\{f(t)\} = \ldots\ldots\ldots\ldots$$

3

$$\boxed{L\{f(t)\} = \bar{F}(s) + e^{-sT}\bar{F}(s) + e^{-2sT}\bar{F}(s) + \cdots}$$

Because

$L\{\bar{f}(t)u(t-c)\} = e^{-sc}L\{\bar{f}(t)\}$ by the second shift theorem.

We can factor out $\bar{F}(s)$ and write $L\{f(t)\}$ as

$$L\{f(t)\} = \left(1 + e^{-sT} + e^{-2sT} + \ldots\right)\bar{F}(s)$$

Now, do you remember the series $1 + x + x^2 + x^3 + \ldots$? This can be written in closed form as

$$1 + x + x^2 + x^3 + \ldots = \ldots\ldots\ldots\ldots$$

4

$$\boxed{1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}}$$

Because

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

either by the binomial theorem or by performing the long division.

So, if we let $x = e^{-sT}$ then

$$1 + e^{-sT} + e^{-2sT} + \ldots = \ldots\ldots\ldots\ldots$$

5

$$\boxed{1 + e^{-sT} + e^{-2sT} + \ldots = \frac{1}{1 - e^{-sT}}}$$

And so the Laplace transform of $f(t)$ is given as

$L\{f(t)\} = \left(1 + e^{-sT} + e^{-2sT} + \ldots\right)\bar{F}(s) = \ldots\ldots\ldots\ldots$ where $\bar{F}(s) = \ldots\ldots\ldots\ldots$

6

$$\boxed{L\{f(t)\} = \frac{1}{(1-e^{-sT})}\bar{F}(s) \text{ where } \bar{F}(s) = \int_0^T e^{-st}f(t)\,dt}$$

Note that we integrate $e^{-st}f(t)$ over one cycle, that is from $t = 0$ to $t = T$, and not from $t = 0$ to $t = \infty$ as we did previously.

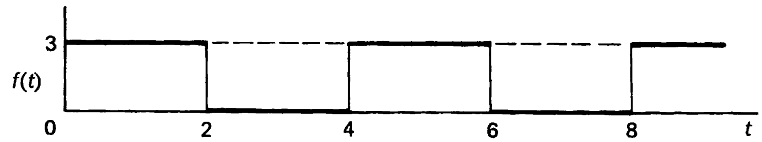
This is an important result. Make a note of it – then we shall apply it

▶

Example 1

Find the Laplace transform of the function $f(t)$ defined by

$$\left.\begin{aligned} f(t) &= 3 \quad 0 < t < 2 \\ &= 0 \quad 2 < t < 4 \end{aligned}\right\} \quad f(t+4) = f(t)$$



The expression for $L\{f(t)\}$ is

.... (do not evaluate it yet)

7

$$\boxed{L\{f(t)\} = \frac{1}{1-e^{-4s}}\int_0^4 e^{-st} \cdot f(t)\, dt}$$

Because the period = 4, i.e. $T = 4$.

The function $f(t) = 3$ for $0 < t < 2$ and $f(t) = 0$ for $2 < t < 4$.

$$\therefore \; L\{f(t)\} = \frac{1}{1-e^{-4s}}\int_0^2 e^{-st} \cdot 3\, dt = \ldots\ldots\ldots\ldots$$

8

$$\boxed{L\{f(t)\} = \frac{3}{s(1+e^{-2s})}}$$

Because

$$\begin{aligned} L\{f(t)\} &= \frac{3}{1-e^{-4s}}\left[\frac{e^{-st}}{-s}\right]_0^2 = \frac{3}{1-e^{-4s}}\left\{\left(\frac{e^{-2s}}{-s}\right) - \left(\frac{1}{-s}\right)\right\} \\ &= \frac{3}{1-e^{-4s}}\left\{\frac{1-e^{-2s}}{s}\right\} = \frac{3}{s(1+e^{-2s})} \end{aligned}$$

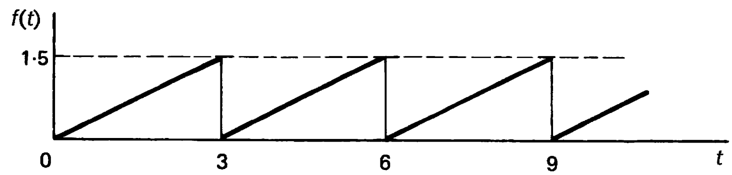
That is all there is to it. Now for another, so move on

9

Example 2

Find the Laplace transform of the periodic function defined by

$$f(t) = t/2 \qquad 0 < t < 3$$
$$f(t+3) = f(t)$$



Because in this case, period = 3, i.e. $T = 3$.

$$\therefore\ L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}\cdot f(t)\,dt$$
$$= \frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\cdot\left(\frac{t}{2}\right)dt$$
$$\therefore\ 2(1-e^{-3s})L\{f(t)\} = \int_0^3 t\cdot e^{-st}\,dt$$

Integrating by parts and simplifying the result gives

$L\{f(t)\} = \ldots\ldots\ldots\ldots$

10

$$\boxed{L\{f(t)\} = \frac{1}{2s^2}\left\{1 - \frac{3s}{e^{3s}-1}\right\}}$$

Because

$$2(1-e^{-3s})L\{f(t)\} = \int_0^3 te^{-st}\,dt$$
$$= \left[t\left(\frac{e^{-st}}{-s}\right)\right]_0^3 + \frac{1}{s}\int_0^3 e^{-st}\,dt$$
$$= -\frac{3e^{-3s}}{s} + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^3$$
$$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}$$
$$\therefore\ L\{f(t)\} = \frac{1}{2s^2}\left\{1 - \frac{3se^{-3s}}{1-e^{-3s}}\right\}$$
$$= \frac{1}{2s^2}\left\{1 - \frac{3s}{e^{3s}-1}\right\}$$

▶

Example 3

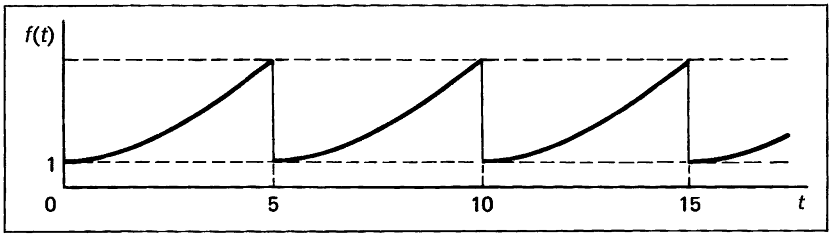
Sketch the graph of the function

$$f(t) = e^t \qquad 0 < t < 5$$
$$f(t+5) = f(t)$$

and determine its Laplace transform.

First we sketch the graph of $f(t)$, which is ....

11



Clearly, period $= 5 \quad \therefore \; T = 5$

$$L\{f(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} \cdot f(t)\,dt \quad \text{gives}$$

$L\{f(t)\} =$ ....

Complete the working

12

$$\boxed{L\{f(t)\} = \frac{1 - e^{-5(s-1)}}{(s-1)(1 - e^{-5s})}}$$

Because

$$L\{f(t)\} = \frac{1}{1 - e^{-5s}} \int_0^5 e^{-st} \cdot e^t\,dt$$

$$\therefore \; (1 - e^{-5s})L\{f(t)\} = \int_0^5 e^{-(s-1)t}\,dt$$

$$= \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^5 = \frac{1}{s-1}\left\{1 - e^{-5(s-1)}\right\}$$

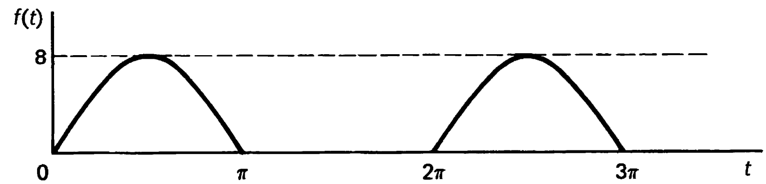
$$\therefore \; L\{f(t)\} = \frac{1 - e^{-5(s-1)}}{(s-1)(1 - e^{-5s})}$$

All very straightforward.

Example 4

Determine the Laplace transform of the half-wave rectifier output waveform defined by

$$\left.\begin{aligned} f(t) &= 8\sin t \quad 0 < t < \pi \\ &= 0 \qquad\quad \pi < t < 2\pi \end{aligned}\right\} \quad f(t + 2\pi) = f(t)$$



Here the period is 2π i.e. $T = 2\pi$.

In general, for a periodic function of period T

$$L\{f(t)\} = \ldots\ldots\ldots\ldots$$

13

$$\boxed{L\{f(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} \cdot f(t)\,\mathrm{d}t}$$

So, for this example

$$L\{f(t)\} = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} \cdot f(t)\,\mathrm{d}t$$

$$\therefore \ (1 - e^{-2\pi s})L\{f(t)\} = \int_0^{\pi} e^{-st} \cdot 8\sin t\,\mathrm{d}t$$

Writing $\sin t$ as the imaginary part of e^{jt}, i.e. $\sin t \equiv \mathscr{I}e^{jt}$,

$$\begin{aligned}(1 - e^{-2\pi s})L\{f(t)\} &= 8\mathscr{I}\int_0^{\pi} e^{-st} \cdot e^{jt}\,\mathrm{d}t \\ &= 8\mathscr{I}\int_0^{\pi} e^{-(s-j)t}\,\mathrm{d}t\end{aligned}$$

and this you can finish off in the usual manner, giving

$$L\{f(t)\} = \ldots\ldots\ldots\ldots$$

14

$$\boxed{L\{f(t)\} = \frac{8}{(s^2+1)(1-e^{-\pi s})}}$$

Because

$$\begin{aligned}(1-e^{-2\pi s})L\{f(t)\} &= 8\cdot\mathscr{I}\int_0^{\pi} e^{-(s-j)t}\,dt\\ &= 8\cdot\mathscr{I}\left[\frac{e^{-(s-j)t}}{-(s-j)}\right]_0^{\pi}\\ &= \mathscr{I}\left\{\frac{-8}{s-j}\left[e^{-(s-j)\pi}-1\right]\right\}\\ &= 8\cdot\mathscr{I}\left\{\frac{1}{s-j}\left[1-e^{-s\pi}e^{j\pi}\right]\right\}\end{aligned}$$

But $e^{j\pi} = \cos\pi + j\sin\pi = -1$.

$$\begin{aligned}\therefore\ (1-e^{-2\pi s})L\{f(t)\} &= 8\cdot\mathscr{I}\left\{\frac{1}{s-j}(1+e^{-s\pi})\right\}\\ &= 8\cdot\mathscr{I}\left\{\frac{s+j}{s^2+1}(1+e^{-\pi s})\right\} = 8\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}\\ \therefore\ L\{f(t)\} &= \frac{1}{1-e^{-2\pi s}}\times 8\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}\\ &= \frac{8}{(1-e^{-\pi s})(s^2+1)}\end{aligned}$$

Now let us consider the corresponding inverse transforms when periodic functions are involved.

15

Inverse transforms

Finding inverse transforms of functions of s which are transforms of periodic functions is not as straightforward as in earlier examples, for the transforms result from integration over one cycle and not from $t = 0$ to $t = \infty$. Hence we have no simple table of inverse transforms upon which to draw.

However, all difficulties can be surmounted and an example will show how we deal with this particular problem.

Example 1

Determine the inverse transform

$$L^{-1}\left\{\frac{2+e^{-2s}-3e^{-s}}{s(1-e^{-2s})}\right\}$$

The first thing we see is the factor $(1-e^{-2s})$ in the denominator, which suggests a periodic function of period 2 units, i.e. $\dfrac{1}{1-e^{-Ts}}$ where $T = 2$.

The key to the solution is to write $(1-e^{-2s})$ in the denominator as $(1-e^{-2s})^{-1}$ in the numerator and to expand this as a binomial series.

We remember that $(1-x)^{-1} = \ldots\ldots\ldots\ldots$

16

$$\boxed{(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots}$$

$$\therefore\ \left(1-e^{-2s}\right)^{-1} = 1 + \left(e^{-2s}\right) + \left(e^{-2s}\right)^2 + \left(e^{-2s}\right)^3 + \dots$$
$$= 1 + e^{-2s} + e^{-4s} + e^{-6s} + \dots$$
$$\therefore\ L\{f(t)\} = \frac{2 + e^{-2s} - 3e^{-s}}{s(1-e^{-2s})} = \frac{1}{s}\left(2 + e^{-2s} - 3e^{-s}\right)\left(1-e^{-2s}\right)^{-1}$$
$$= \frac{1}{s}\left(2 + e^{-2s} - 3e^{-s}\right)\left(1 + e^{-2s} + e^{-4s} + e^{-6s} + e^{-8s} + \dots\right)$$

We now multiply the second series by each term of the first in turn and collect up like terms, giving

$$L\{f(t)\} = \frac{1}{s}\left\{\begin{array}{lllllll} 2 & & +2e^{-2s} & & +2e^{-4s} & & +2e^{-6s}\ \dots \\ & & +\ e^{-2s} & & +\ e^{-4s} & & +\ e^{-6s}\ \dots \\ & -3e^{-s} & & -3e^{-3s} & & -3e^{-5s} & \dots \end{array}\right\}$$
$$= \dots\dots\dots\dots$$

17

$$\boxed{L\{f(t)\} = \frac{1}{s}\{2 - 3e^{-s} + 3e^{-2s} - 3e^{-3s} + 3e^{-4s} - 3e^{-5s} + \dots\}}$$

Each term is of the form $\dfrac{e^{-cs}}{s}$, so, expressing $f(t)$ in unit step form, we have

$$f(t) = \dots\dots\dots\dots$$

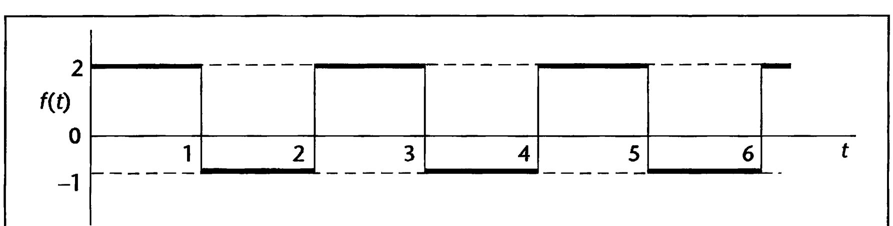
18

$$\boxed{f(t) = 2u(t) - 3u(t-1) + 3u(t-2) - 3u(t-3) + 3u(t-4)\dots}$$

and from this we can sketch the waveform, which is therefore

$$\dots\dots\dots\dots$$

19



We can finally define this periodic function in analytical terms.

$$f(t) = \dots\dots\dots\dots$$

20

$$\boxed{\left.\begin{aligned} f(t) &= 2 \quad 0 < t < 1 \\ &= -1 \quad 1 < t < 2 \end{aligned}\right\} f(t+2) = f(t)}$$

The key to the whole process is thus to ....

21

> express $(1 - e^{-Ts})$ in the denominator as $(1 - e^{-Ts})^{-1}$ in the numerator and to expand this as a binomial series.

We do this by making use of the basic series

$$(1 - x)^{-1} = \ldots\ldots\ldots\ldots$$

22

$$\boxed{(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots}$$

Example 2

Determine $L^{-1}\left\{\dfrac{3(1 - e^{-s})}{s(1 - e^{-3s})}\right\}$ and sketch the resulting waveform of $f(t)$.

$$L\{f(t)\} = \frac{3}{s}(1 - e^{-s})(1 - e^{-3s})^{-1}$$

$$= \ldots\ldots\ldots\ldots \qquad \text{(next step)}$$

23

$$\boxed{L\{f(t)\} = \frac{3}{s}(1 - e^{-s})(1 + e^{-3s} + e^{-6s} + e^{-9s} + \ldots)}$$

which multiplied out gives

$$L\{f(t)\} = \frac{3}{s}(1 - e^{-s} + e^{-3s} - e^{-4s} + e^{-6s} - e^{-7s} + \ldots)$$

$$= \frac{3}{s} - \frac{3e^{-s}}{s} + \frac{3e^{-3s}}{s} - \frac{3e^{-4s}}{s} + \frac{3e^{-6s}}{s} - \ldots$$

And in unit step form, this gives

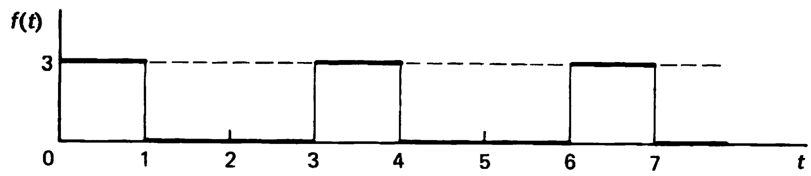
$$f(t) = \ldots\ldots\ldots\ldots$$

24

$$\boxed{f(t) = 3u(t) - 3u(t-1) + 3u(t-3) - 3u(t-4) + \ldots}$$

The waveform is thus ....

25



$$\left.\begin{array}{ll} f(t)=3 & 0<t<1 \\ f(t)=0 & 1<t<3 \end{array}\right\} f(t+3)=f(t)$$

And now, one more. They are all done in the same way

26

Example 3

If $L\{f(t)\}=\dfrac{1}{2s^2}-\dfrac{2e^{-4s}}{s(1-e^{-4s})}$, determine $f(t)$ and sketch the waveform.

The first term is easy enough. In unit step form $L^{-1}\left\{\dfrac{1}{2s^2}\right\}=\dfrac{t}{2}\cdot u(t)$

From the second term

$$\begin{aligned}\frac{2e^{-4s}}{s(1-e^{-4s})}&=\frac{2}{s}\left\{e^{-4s}(1-e^{-4s})^{-1}\right\}\\&=\frac{2}{s}\{e^{-4s}(1+e^{-4s}+e^{-8s}+e^{-12s}+\ldots)\}\\&=\frac{2e^{-4s}}{s}+\frac{2e^{-8s}}{s}+\frac{2e^{-12s}}{s}+\frac{2e^{-16s}}{s}+\ldots\end{aligned}$$

$\therefore\ f(t)=\ldots\ldots\ldots\ldots$ (in unit step form)

27

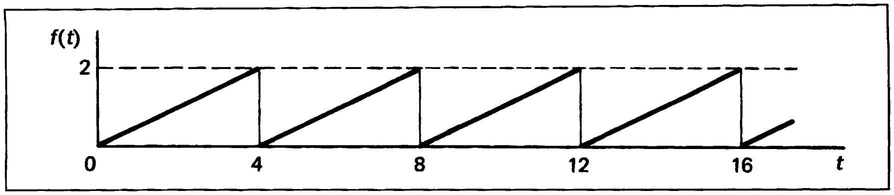
$$\boxed{f(t)=\frac{t}{2}\cdot u(t)-2u(t-4)-2u(t-8)-2u(t-12)-\ldots}$$

Now we have to draw the waveform. Consider the function terms up to each break point in turn.

$$\begin{array}{lll} 0<t<4 & f(t)=\dfrac{t}{2} & f(0)=0;\ \ f(4)=2 \\ 4<t<8 & f(t)=\dfrac{t}{2}-2 & f(4)=0;\ \ f(8)=2 \\ 8<t<12 & f(t)=\dfrac{t}{2}-2-2 & f(8)=0;\ \ f(12)=2 \text{ etc.} \end{array}$$

So the waveform is $\ldots\ldots\ldots\ldots$

28



Expressed analytically, we finally have

$$f(t) = \frac{t}{2} \quad 0 < t < 4, \quad f(t+4) = f(t)$$

The Dirac delta – the unit impulse

29

So far we have dealt with a number of standard Laplace transforms and then the Heaviside unit step function with some of its applications. We now come to consider an entity that is different from any of the functions we have used before because it is not a proper function. Rather than being defined by its inputs and corresponding outputs it is defined by its effect on other functions. If $f(t)$ represents a function then the Dirac delta $\delta(t)$ is defined by the integral

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a)$$

$\delta(t)$ is often referred to as the **Dirac delta function** even though it is not a function in the conventional sense of being completely defined in terms of its outputs for the corresponding inputs. The nearest that can be achieved in defining it in function terms is

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$$

From the definition, if $f(t) = 1$ then

$$\int_{-\infty}^{\infty} \delta(t-a)\,\mathrm{d}t = \ldots\ldots\ldots\ldots$$

30

$$\boxed{\int_{-\infty}^{\infty} \delta(t-a)\,\mathrm{d}t = 1}$$

Because

$\displaystyle\int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a)$ and $f(t) = 1$ so $f(a) = 1$, therefore

$\displaystyle\int_{-\infty}^{\infty} \delta(t-a)\,\mathrm{d}t = 1$ hence the name *unit impulse.*

Also, if $p < a < q$ then

$$\int_{p}^{q} \delta(t-a)\,\mathrm{d}t = \ldots\ldots\ldots\ldots$$

31

$$\boxed{\int_p^q \delta(t-a)\,dt = 1}$$

Because

$$\int_{-\infty}^{\infty} \delta(t-a)\,dt = \int_{-\infty}^{p} \delta(t-a)\,dt + \int_p^q \delta(t-a)\,dt + \int_q^{\infty} \delta(t-a)\,dt$$

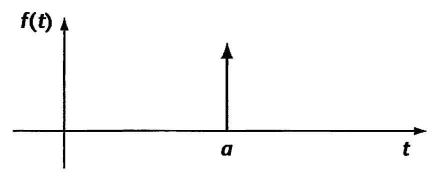
$$= 0 + \int_p^q \delta(t-a)\,dt + 0 \quad \text{since } \delta(t-a) = 0 \text{ for } -\infty < t \leq p \text{ and } q \leq t < \infty$$

$$= 1$$

So that $\displaystyle\int_p^q \delta(t-a)\,dt = 1$

Graphical representation

Graphically the Dirac delta or unit impulse $\delta(t-a)$ is represented by the horizontal axis with a vertical line of infinite length at $t = a$.



32

So far, then, we have

(a) $\displaystyle\int_p^q \delta(t-a)\,dt = 1$

(b) $\displaystyle\int_p^q f(t)\cdot\delta(t-a)\,dt = f(a)$

provided, in each case, that $p < a < q$.

Example 1

To evaluate $\displaystyle\int_1^3 (t^2+4)\cdot\delta(t-2)\,dt$.

The factor $\delta(t-2)$ shows that the impulse occurs at $t = 2$, i.e. $a = 2$.

$f(t) = t^2 + 4 \qquad \therefore\ f(a) = f(2) = 4 + 4 = 8$

$\therefore\ \displaystyle\int_1^3 (t^2+4)\cdot\delta(t-2)\,dt = f(2) = 8$

Example 2

To evaluate $\int_0^{\pi} \cos 6t \cdot \delta(t - \pi/2)\,dt$.

$$\int_0^{\pi} \cos 6t \cdot \delta(t - \pi/2)\,dt = f(\pi/2) = \cos 3\pi = -1$$

and in the same way

(a) $\int_0^{6} 5 \cdot \delta(t-3)\,dt = \ldots\ldots\ldots\ldots$

(b) $\int_2^{5} e^{-2t} \cdot \delta(t-4)\,dt = \ldots\ldots\ldots\ldots$

(c) $\int_0^{\infty} (3t^2 - 4t + 5) \cdot \delta(t-2)\,dt = \ldots\ldots\ldots\ldots$

33

(a) $\int_0^{6} 5 \cdot \delta(t-3)\,dt = 5 \times 1 = 5$

(b) $\int_2^{5} e^{-2t} \cdot \delta(t-4)\,dt = f(4) = \left[e^{-2t}\right]_{t=4} = e^{-8}$

(c) $\int_0^{\infty} (3t^2 - 4t + 5) \cdot \delta(t-2)\,dt = 12 - 8 + 5 = 9$

Nothing could be easier. It all rests on the fact that, provided $p < a < q$

$$\int_p^{q} f(t) \cdot \delta(t-a)\,dt = \ldots\ldots\ldots\ldots$$

34

$$\boxed{f(a)}$$

Now let us consider the Laplace transform of $\delta(t-a)$.

On then to the next frame

35

Laplace transform of $\delta(t-a)$

We have already shown that

$$\int_p^{q} f(t) \cdot \delta(t-a)\,dt = f(a) \qquad p < a < q$$

Therefore, if $p = 0$ and $q = \infty$

$$\int_0^{\infty} f(t) \cdot \delta(t-a)\,dt = f(a)$$

Hence, if $f(t) = e^{-st}$, this becomes

$$\int_0^{\infty} e^{-st} \cdot \delta(t-a)\,dt = L\{\delta(t-a)\}$$

$$= \ldots\ldots\ldots\ldots$$

36

$\boxed{e^{-as}}$

i.e. the value of $f(t)$, i.e. e^{-st}, at $t = a$.

$L\{\delta(t-a)\} = e^{-as}$

It follows from this that the Laplace transform of the impulse function at the origin is ....

37

$\boxed{1}$

Because, for $a = 0$, $L\{\delta(t-a)\} = L\{\delta(t)\} = e^0 = 1$

$$\therefore\ L\{\delta(t)\} = 1$$

Finally, let us deal with the more general case of $L\{f(t)\cdot\delta(t-a)\}$.

We have $L\{f(t)\cdot\delta(t-a)\} = \int_0^\infty e^{-st}\cdot f(t)\cdot\delta(t-a)\,\mathrm{d}t$. Now the integrand $e^{-st}\cdot f(t)\cdot\delta(t-a) = 0$ for all values of t except at $t = a$ at which point $e^{-st} = e^{-as}$, and $f(t) = f(a)$.

$$\therefore\ L\{f(t)\cdot\delta(t-a)\} = f(a)\cdot e^{-as}\int_0^\infty \delta(t-a)\,\mathrm{d}t$$
$$= f(a)\cdot e^{-as}(1)$$
$$\therefore\ L\{f(t)\cdot\delta(t-a)\} = f(a)e^{-as}$$

Another important result to note. Then let us deal with some examples

38

We have $L\{f(t)\cdot\delta(t-a)\} = f(a)\cdot e^{-as}$

Therefore

(a) $L\{6\cdot\delta(t-4)\}$ $\quad a = 4,\quad \therefore\ L\{6\cdot\delta(t-4)\} = 6e^{-4s}$

(b) $L\{t^3\cdot\delta(t-2)\}$ $\quad a = 2,\quad \therefore\ L\{t^3\cdot\delta(t-2)\} = 8e^{-2s}$

Similarly

(c) $L\{\sin 3t\cdot\delta(t-\pi/2)\} =$ ....

39

$\boxed{-e^{-\pi s/2}}$

Because

$$L\{\sin 3t\cdot\delta(t-\pi/2)\} = [\sin 3t]_{t=\pi/2}\cdot e^{-\pi s/2} = -e^{-\pi s/2}$$

and

(d) $L\{\cosh 2t\cdot\delta(t)\} =$ ....

40

$$\boxed{1}$$

Because

$$L\{\cosh 2t \cdot \delta(t)\} = [\cosh 2t]_{t=0} \cdot e^0 = \cosh 0 \cdot (1) = 1$$

So our main conclusions so far are as follows.

(1) $\int_p^q \delta(t-a)\,dt = \ldots\ldots\ldots\ldots$ provided $\ldots\ldots\ldots\ldots$

(2) $\int_p^q f(t) \cdot \delta(t-a)\,dt = \ldots\ldots\ldots\ldots$ provided $\ldots\ldots\ldots\ldots$

(3) $L\{\delta(t-a)\} = \ldots\ldots\ldots\ldots$

(4) $L\{\delta(t)\} = \ldots\ldots\ldots\ldots$

(5) $L\{f(t) \cdot \delta(t-a)\} = \ldots\ldots\ldots\ldots$

41

(1) $\int_p^q \delta(t-a)\,dt = 1$ provided $p < a < q$

(2) $\int_p^q f(t) \cdot \delta(t-a)\,dt = f(a)$ provided $p < a < q$

(3) $L\{\delta(t-a)\} = e^{-as}$

(4) $L\{\delta(t)\} = 1$

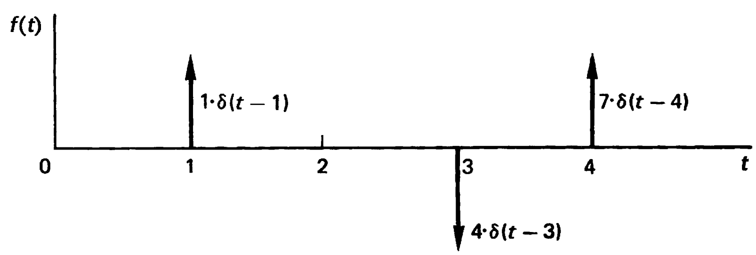
(5) $L\{f(t) \cdot \delta(t-a)\} = f(a) \cdot e^{-as}$

Just check that you have noted this important list – the basis of all work on the Dirac delta function.

Now for one further example on this section

Example

Impulses of 1, 4, 7 units occur at $t = 1$, $t = 3$ and $t = 4$ respectively, in the directions shown.



Write down an expression for $f(t)$ and determine its Laplace transform.

We have $f(t) = 1 \cdot \delta(t-1) - 4 \cdot \delta(t-3) + 7 \cdot \delta(t-4)$.

Then $L\{f(t)\} = \ldots\ldots\ldots\ldots$

$$\boxed{L\{f(t)\} = e^{-s} - 4e^{-3s} + 7e^{-4s}}$$

and that is all there is to that.

The derivative of the unit step function

One further consideration is interesting.

Consider some function $f(t)$ that is zero outside some finite interval $[a, b]$ of the real line. That is, $f(t) = 0$ for $t < a$ and $t > b$, then

$$\int_{-\infty}^{\infty} [u(t)f(t)]' \,\mathrm{d}t = [u(t)f(t)]_{-\infty}^{\infty} = 0$$

where $u(t)$ is the unit step function and $f(t)$ is zero at the limits. Now

$$\int_{-\infty}^{\infty} [u(t)f(t)]' \,\mathrm{d}t = \int_{-\infty}^{\infty} u'(t)f(t)\,\mathrm{d}t + \int_{-\infty}^{\infty} u(t)f'(t)\,\mathrm{d}t$$

and so

$$\int_{-\infty}^{\infty} u'(t)f(t)\,\mathrm{d}t = -\int_{-\infty}^{\infty} u(t)f'(t)\,\mathrm{d}t$$

This means that

$$\begin{aligned}
\int_{-\infty}^{\infty} u'(t)f(t)\,\mathrm{d}t &= -\int_{-\infty}^{\infty} u(t)f'(t)\,\mathrm{d}t \\
&= -\int_{0}^{\infty} f'(t)\,\mathrm{d}t && \text{Because the unit step is zero for negative } t \\
&= -\Big[f(t)\Big]_0^{\infty} \\
&= -f(\infty) + f(0) \\
&= f(0) && \text{Because } f(\infty) = 0 \text{ by definition} \\
&= \int_{-\infty}^{\infty} \delta(t)f(t)\,\mathrm{d}t && \text{By the definition of the Dirac delta}
\end{aligned}$$

and so $u'(t) = \delta(t)$ – *the unit impulse is equal to the derivative of the unit step function.*

Differential equations involving the unit impulse

43

Example 1

A system has the equation of motion

$$\ddot{x} + 6\dot{x} + 8x = g(t)$$

where $g(t)$ is an impulse of 4 units applied at $t = 5$. At $t = 0$, $x = 0$ and $\dot{x} = 3$. Determine an expression for the displacement x in terms of t.

The impulse of 4 units is applied at $t = 5$. $\therefore\ g(t) = 4 \cdot \delta(t-5)$.

$$\therefore\ \ddot{x} + 6\dot{x} + 8x = 4 \cdot \delta(t-5) \quad \text{At } t = 0,\ x = 0,\ \dot{x} = 3.$$

Taking Laplace transforms this differential equation becomes

....

44

$$\boxed{(s^2\bar{x} - sx_0 - x_1) + 6(s\bar{x} - x_0) + 8\bar{x} = 4e^{-5s}}$$

Now $x_0 = 0$; $x_1 = 3$

$$\therefore\ s^2\bar{x} - 3 + 6s\bar{x} + 8\bar{x} = 4e^{-5s}$$

$$\therefore\ (s^2 + 6s + 8)\bar{x} = 3 + 4e^{-5s}$$

$$\therefore\ \bar{x} = \left(3 + 4e^{-5s}\right)\frac{1}{(s+2)(s+4)}$$

Writing $\dfrac{1}{(s+2)(s+4)}$ in partial fractions, we get

$\bar{x} =$ ....

45

$$\boxed{\bar{x} = \left(3 + 4e^{-5s}\right)\left\{\frac{1}{2} \cdot \frac{1}{s+2} - \frac{1}{2} \cdot \frac{1}{s+4}\right\}}$$

$$\therefore\ \bar{x} = \frac{3}{2}\left\{\frac{1}{s+2} - \frac{1}{s+4}\right\} + 2\left\{\frac{e^{-5s}}{s+2} - \frac{e^{-5s}}{s+4}\right\}$$

Taking inverse transforms

$$x = \frac{3}{2}\left\{e^{-2t} - e^{-4t}\right\} + 2\left\{e^{-2(t-5)} \cdot u(t-5) - e^{-4(t-5)} \cdot u(t-5)\right\}$$

$$= \frac{3}{2}\left\{e^{-2t} - e^{-4t}\right\} + 2\left\{e^{-2t} \cdot e^{10} \cdot u(t-5) - e^{-4t} \cdot e^{20} \cdot u(t-5)\right\}$$

which simplifies to $x =$ ....

46

$$x = e^{-2t}\left\{\frac{3}{2} + 2e^{10} \cdot u(t-5)\right\} - e^{-4t}\left\{\frac{3}{2} + 2e^{20} \cdot u(t-5)\right\}$$

Example 2

Solve the equation $\ddot{x} + 4\dot{x} + 13x = 2 \cdot \delta(t)$ where, at $t = 0$, $x = 2$ and $\dot{x} = 0$.

$\ddot{x} + 4\dot{x} + 13x = 2 \cdot \delta(t) \qquad x_0 = 2;\ x_1 = 0$

Expressing in Laplace transforms, we have

....

47

$$(s^2\bar{x} - sx_0 - x_1) + 4(s\bar{x} - x_0) + 13\bar{x} = 2 \cdot (1)$$

Inserting the initial conditions and simplifying,

$\bar{x} =$ ....

48

$$\bar{x} = (2s + 10)\frac{1}{s^2 + 4s + 13}$$

Rearranging the denominator by completing the square, this can be written

$$\bar{x} = (2s + 10)\frac{1}{(s+2)^2 + 9}$$

$\therefore\ x =$ ....

49

$$x = 2e^{-2t}\{\cos 3t + \sin 3t\}$$

Because

$$\bar{x} = \frac{2(s+2)}{(s+2)^2 + 9} + \frac{6}{(s+2)^2 + 9}$$

$\therefore\ x = 2e^{-2t}\cos 3t + 2e^{-2t}\sin 3t$

$\therefore\ x = 2e^{-2t}\{\cos 3t + \sin 3t\}$

Now for one further example for you to work through on your own.

So move on

50

Example 3

The equation of motion of a system is

$\ddot{x} + 5\dot{x} + 4x = g(t)$ where $g(t) = 3 \cdot \delta(t-2)$.

At $t = 0$, $x = 2$ and $\dot{x} = -2$. Determine an expression for the displacement x in terms of t.

We have $\ddot{x} + 5\dot{x} + 4x = 3 \cdot \delta(t-2)$ with $x_0 = 2$ and $x_1 = -2$.

As before, you can express this in Laplace transforms, substitute the initial conditions, simplify to obtain an expression for x and finally take inverse transforms to determine the required expression for x.

Work right through it carefully. It is good revision and there are no snags.

$$x = \dots\dots\dots\dots$$

51

$$\boxed{x = e^{-t}\{2 + e^2 \cdot u(t-2)\} - e^8 \cdot e^{-4t} \cdot u(t-2)}$$

Here is the working for you to check.

$$\ddot{x} + 5\dot{x} + 4x = 3 \cdot \delta(t-2) \text{ with } x_0 = 2 \text{ and } x_1 = -2$$

$$(s^2\bar{x} - sx_0 - x_1) + 5(s\bar{x} - x_0) + 4\bar{x} = 3e^{-2s}$$

$$s^2\bar{x} - 2s + 2 + 5s\bar{x} - 10 + 4\bar{x} = 3e^{-2s}$$

$$(s^2 + 5s + 4)\bar{x} - 2s - 8 = 3e^{-2s}$$

$$\therefore\ (s+1)(s+4)\bar{x} = 2s + 8 + 3e^{-2s}$$

$$\therefore\ \bar{x} = \frac{2(s+4)}{(s+1)(s+4)} + e^{-2s} \cdot \frac{3}{(s+1)(s+4)}$$

$$= \frac{2}{s+1} + e^{-2s}\left\{\frac{1}{s+1} - \frac{1}{s+4}\right\}$$

$$\therefore\ \bar{x} = \frac{2}{s+1} + \frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s+4}$$

$$\therefore\ x = 2e^{-t} + u(t-2) \cdot e^{-(t-2)} - u(t-2) \cdot e^{-4(t-2)}$$

$$= 2e^{-t} + u(t-2) \cdot e^2 \cdot e^{-t} - u(t-2) \cdot e^8 \cdot e^{-4t}$$

$$x = e^{-t}\{2 + e^2 \cdot u(t-2)\} - e^8 \cdot e^{-4t} \cdot u(t-2)$$

Harmonic oscillators

52

If the position of a system at time t is described by the expression $f(t)$ where $f(t)$ satisfies the differential equation

$$af''(t) + bf(t) = 0,\ f(0) = \alpha \text{ and } f'(0) = \beta$$

(and where a and b have the same sign)

then, taking Laplace transforms of both sides gives

$$L\{af''(t) + bf(t)\} = L\{0\}$$

That is

$$a\left[s^2F(s) - s\alpha - \beta\right] + b\left[F(s)\right] = 0$$

Collecting like terms gives

$$\left(as^2 + b\right)F(s) = s\alpha + \beta$$

giving

$$F(s) = \frac{s\alpha + \beta}{as^2 + b}$$

Therefore $F(s) = \dfrac{s(\alpha/a)}{s^2 + (b/a)} + \dfrac{\beta/a}{s^2 + (b/a)}$ and so

$$f(t) = \frac{\alpha}{a}\cos\sqrt{\frac{b}{a}}t + \frac{\beta}{a}\sin\sqrt{\frac{b}{a}}t$$

The system executes *simple harmonic, oscillatory motion with frequency* $\sqrt{\dfrac{b}{a}}$ radians per unit of time and with period $\dfrac{2\pi}{\sqrt{b/a}} = 2\pi\sqrt{\dfrac{a}{b}}$. It is called an **harmonic oscillator**. Let's try some examples.

Example 1

Find the solution to the harmonic oscillator

$$f''(t) + 16f(t) = 0 \text{ where } f(0) = 1 \text{ and } f'(0) = 0$$

Taking Laplace transforms gives

$$F(s) = \dots\dots\dots\dots$$

53

$$\boxed{F(s) = \frac{s}{s^2 + 16}}$$

Because

Taking Laplace transforms $L\{f''(t) + 16f(t)\} = L\{0\}$.

That is $s^2F(s) - s + 16F(s) = 0$ and so

$$F(s) = \frac{s}{s^2 + 16}$$

This means that

$$f(t) = \dots\dots\dots\dots$$

54

$$\boxed{f(t) = \cos 4t}$$

Because

$F(s) = \dfrac{s}{s^2+16} = \dfrac{s}{s^2+4^2}$ so $f(t) = \cos 4t$ from the Table of Laplace transforms on page 68.

The motion of this system is then periodic with frequency 4 radians per unit of time and with period $2\pi/4 = \pi/2$ units of time.

Example 2

The frequency and period of the harmonic oscillator whose position $f(t)$ satisfies the differential equation

$5f''(t) + 10f(t) = 0$ where $f(0) = 0$ and $f'(0) = 4$

is given as

frequency .... radians per unit of time
and period .... units of time

55

$$\boxed{\text{frequency } \sqrt{2} \text{ and period } \sqrt{2}\pi}$$

Because

Taking Laplace transforms gives

$L\{5f''(t) + 10f(t)\} = L\{0\}$ that is $5s^2F(s) - 4 + 10F(s) = 0$ so that

$$F(s) = \frac{4}{5s^2+10} = \frac{4/5}{s^2+2}$$

and from the Table of Laplace transforms on page 68

$$f(t) = \frac{2\sqrt{2}}{5}\sin\sqrt{2}t$$

This is periodic with frequency $\sqrt{2}$ radians per unit of time and period $2\pi/\sqrt{2} = \sqrt{2}\pi$ units of time.

Notice that the amplitude of the motion is $\dfrac{2\sqrt{2}}{5}$.

56

Damped motion

Consider the equation

$5f''(t) + 5f'(t) + 10f(t) = 0$ where $f(0) = 0$ and $f'(0) = 4$

This is the same as the last equation in Frame 54 with an extra term added, namely $5f'(t)$. This term describes a particular effect on the system as you will see from the solution.

Solving the differential equation gives

$f(t) = \ldots\ldots\ldots\ldots$

57

$$\boxed{f(t) = \frac{8}{5\sqrt{7}} e^{-t/2} \sin\left(\sqrt{7}t/2\right)}$$

Because

Taking Laplace transforms gives

$L\{5f''(t) + 5f'(t) + 10f(t)\} = L\{0\}$ that is

$5\left(s^2F(s) - 4\right) + 5sF(s) + 10F(s) = 0$

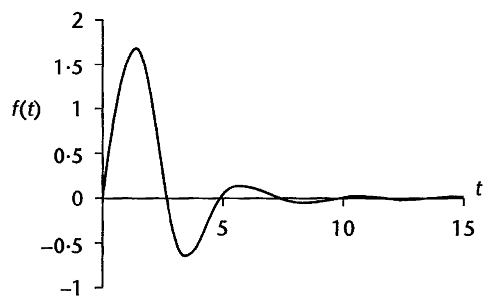
so that

$$F(s) = \frac{20}{5s^2 + 5s + 10} = \frac{4}{s^2 + s + 2} = \frac{4}{(s + 1/2)^2 + \left(\sqrt{7}/2\right)^2}$$

and from the Table of Laplace transforms on page 68

$$f(t) = \frac{8}{\sqrt{7}} e^{-t/2} \sin\left(\sqrt{7}t/2\right)$$

This is periodic with frequency 1 radian per unit of time and period 2π units of time but with an amplitude that is decreasing with time. The graph of this function is as follows



The effect of the $5f'(t)$ in the differential equation is to introduce **damping** into the oscillatory motion so causing the oscillations to decay. Let's try another example.

Example 3

Consider the equation

$5f''(t) + f'(t) + 10f(t) = 0$ where $f(0) = 0$ and $f'(0) = 4$

This equation is again similar to the previous equation but with a smaller damping term of $f'(t)$ instead of $5f'(t)$. Then here

$$f(t) = \ldots\ldots\ldots\ldots$$

58

$$f(t) = \frac{4}{\sqrt{1 \cdot 99}} e^{-0.1t} \sin \sqrt{1 \cdot 99}t$$

Because

Taking Laplace transforms gives

$L\{5f''(t) + f'(t) + 10f(t)\} = L\{0\}$ that is

$5(s^2F(s) - 4) + sF(s) + 10F(s) = 0$

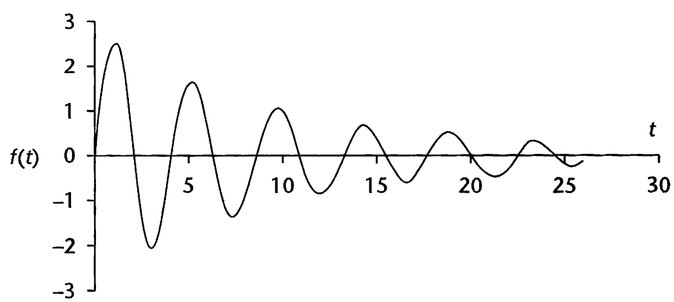
so that

$$F(s) = \frac{20}{5s^2 + 1s + 10} = \frac{4}{s^2 + 0 \cdot 2s + 2} = \frac{4}{(s + 0 \cdot 1)^2 + 1 \cdot 99}$$

and from the Table of Laplace transforms on page 68

$$f(t) = \frac{4}{\sqrt{1 \cdot 99}} e^{-0.1t} \sin \sqrt{1 \cdot 99}t$$

This is periodic with frequency $\sqrt{1 \cdot 99}$ radians per unit of time and period $2\pi/\sqrt{1 \cdot 99}$ units of time and with an amplitude that is decreasing with time. The graph of this function is as follows



Again, the effect of the $f'(t)$ in the differential equation is to introduce damping into the oscillatory motion so causing it to decay. Also because the coefficient of $f'(t)$ is smaller in this example, the damping is less severe.

Forced harmonic motion with damping

59

The equation

$f''(t) + f'(t) + f(t) = e^t$ where $f(0) = 0$ and $f'(0) = 0$

we know would represent damped harmonic motion were it not for the exponential on the right-hand side. To see the effect of the exponential we solve the equation.

Taking Laplace transforms we see that

$$F(s) = \dots\dots\dots\dots$$

60

$$\boxed{F(s) = \frac{1}{(s-1)(s^2+s+1)}}$$

Because

$L\{f''(t) + f'(t) + f(t)\} = L\{e^t\}$ that is $(s^2+s+1)F(s) = \dfrac{1}{s-1}$ so

$$F(s) = \frac{1}{(s-1)(s^2+s+1)}$$

Separating into partial fractions gives

$$F(s) = \dots\dots\dots\dots$$

61

$$\boxed{F(s) = \frac{1}{3(s-1)} - \frac{s+2}{3(s^2+s+1)}}$$

Because

$$\begin{aligned}\frac{1}{(s-1)(s^2+s+1)} &= \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+s+1)}\\ &= \frac{A(s^2+s+1)+(Bs+C)(s-1)}{(s-1)(s^2+s+1)}\end{aligned}$$

Equating numerators and then comparing coefficients of powers of s gives

$1 = A(s^2+s+1) + (Bs+C)(s-1)$

$[s^2]$:	$0 = A + B$	(1)	So (2) + (3):	$1 = 2A - B$
$[s]$:	$0 = A - B + C$	(2)	$2 \times (1)$:	$0 = 2A + 2B$
[CT]:	$1 = A - C$	(3)	Therefore:	$-1 = 3B$

so $B = -1/3 = -A$ and $C = -2/3$

Thus $F(s) = \dfrac{1}{(s-1)(s^2+s+1)} = \dfrac{1}{3(s-1)} - \dfrac{s+2}{3(s^2+s+1)}$

Consequently

$$f(t) = \dots\dots\dots\dots$$

62

$$\boxed{f(t) = \frac{e^t}{3} - \frac{1}{3}e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)}$$

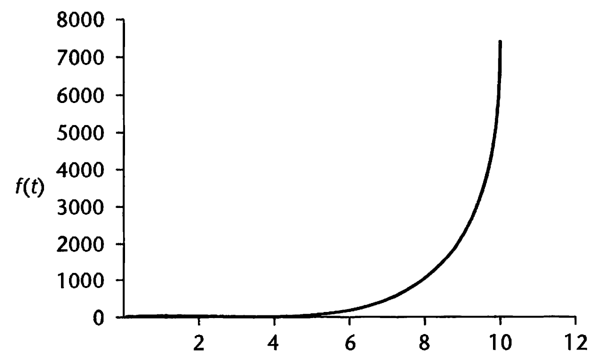
Because

$$\begin{aligned} F(s) &= \frac{1}{3(s-1)} - \frac{s+2}{3(s^2+s+1)} \\ &= \frac{1}{3(s-1)} - \frac{s+\frac{1}{2}}{3\left(\left(s+\frac{1}{2}\right)^2+\frac{3}{4}\right)} - \frac{\frac{3}{2}}{3\left(\left(s+\frac{1}{2}\right)^2+\frac{3}{4}\right)} \end{aligned}$$

So

$$f(t) = \frac{e^t}{3} - \frac{1}{3}e^{-t/2}\left(\cos\tfrac{\sqrt{3}}{2}t + \sqrt{3}\sin\tfrac{\sqrt{3}}{2}t\right)$$

from the Table of Laplace transforms on page 68.



Notice that the term $\frac{1}{3}e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)$ represents damped harmonic motion and is called the **transient** term whereas the term $\frac{e^t}{3}$ represents a **steady-state** term, so called because as the transient term decays the steady-state term remains the dominant part of the solution. The steady-state solution is a direct consequence of the term on the right-hand side of the differential equation.

Try another one for yourself. The transient and steady-state terms of the system described by the differential equation

$$f''(t) + 2f'(t) + 5f(t) = e^{2t} \text{ where } f(0) = 0 \text{ and } f'(0) = 1$$

are Transient term .... Steady-state term ....

63

$$\boxed{-\frac{1}{13}e^{-t}\cos 2t+\frac{5}{13}e^{-t}\sin 2t,\ \frac{1}{13}e^{2t}}$$

Because

Taking Laplace transforms, $L\{f''(t)+2f'(t)+5f(t)\}=L\{e^{2t}\}$. That is

$[s^2F(s)-1]+2sF(s)+5F(s)=\frac{1}{s-2}$, that is

$(s^2+2s+5)F(s)=1+\frac{1}{s-2}=\frac{s-1}{s-2}$

So that $F(s)=\frac{s-1}{(s-2)(s^2+2s+5)}=\frac{A}{s-2}+\frac{Bs+C}{s^2+2s+5}$. Hence

$s-1=A(s^2+2s+5)+(Bs+C)(s-2)$. Equating powers of s gives

$[s^2]$: $\quad 0=A+B$

$[s]$: $\quad 1=2A-2B+C$

[CT]: $\quad -1=5A-2C$

Solving these three equations gives $A=1/13$, $B=-1/13$ and $C=9/13$ so that

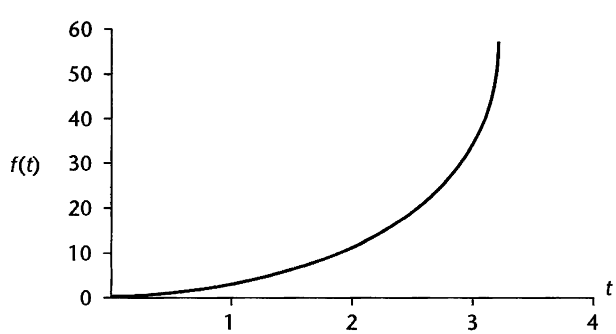
$$F(s)=\frac{1}{13(s-2)}-\frac{s-9}{13(s^2+2s+5)}$$

$$=\frac{1}{13(s-2)}-\frac{s-9}{13\left((s+1)^2+2^2\right)}.$$ That is

$$F(s)=\frac{1}{13(s-2)}-\frac{s+1}{13\left((s+1)^2+2^2\right)}+\frac{10}{13\left((s+1)^2+2^2\right)}$$

Therefore

$$f(t)=\frac{1}{13}e^{2t}-\frac{1}{13}e^{-t}\cos 2t+\frac{5}{13}e^{-t}\sin 2t$$



Next frame

64

Resonance

These differential equations with a function on the right-hand side are called **inhomogeneous differential equations**. They represent systems whose behaviour $f(t)$ is dictated by the structure of the left-hand side and the **forcing function** on the right-hand side. If an undamped and unforced system which exhibits periodic behaviour has a periodic forcing function applied that has the same period then **resonance** will occur and the system will undergo periodic behaviour with an increasing amplitude. An example will illustrate this.

The differential equation

$$f''(t) + f(t) = 0 \text{ where } f(0) = 0 \text{ and } f'(0) = 1$$

represents an undamped, unforced system with behaviour

$$f(t) = \dots\dots\dots\dots$$

65

$$\boxed{f(t) = \sin t}$$

Because

Taking the Laplace transform of both sides of the equation gives

$L\{f''(t) + f(t)\} = L\{0\}$ that is $s^2F(s) - 1 + F(s) = 0$ so that

$F(s) = \dfrac{1}{s^2+1}$ giving $f(t) = \sin t$

If the forcing term $-2\sin t$ is applied to the right-hand side of the equation it has the same period as the natural frequency of the system being forced and so resonance will set in. The differential equation to solve is then

$$f''(t) + f(t) = -2\sin t \text{ where } f(0) = 0 \text{ and } f'(0) = 1$$

This has the solution $f(t) = \dots\dots\dots\dots$

66

$$\boxed{f(t) = t\cos t}$$

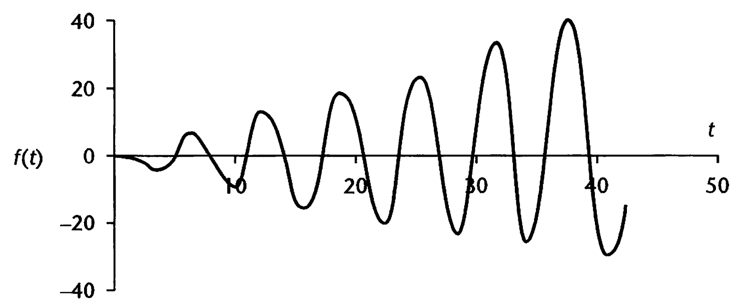
Because

Taking the Laplace transform of both sides of the equation gives

$L\{f''(t) + f(t)\} = L\{-2\sin t\}$ that is $s^2F(s) - 1 + F(s) = -\dfrac{2}{s^2+1}$

so that $F(s) = \dfrac{1}{s^2+1} - \dfrac{2}{(s^2+1)^2}$ giving $F(s) = \dfrac{s^2-1}{(s^2+1)^2}$. Now, the Laplace transform of $\cos t$ is $\dfrac{s}{s^2+1}$ and $\left(\dfrac{s}{s^2+1}\right)' = -\dfrac{s^2-1}{(s^2+1)^2}$.

Therefore $f(t) = t\cos t$



The system undergoes periodic behaviour with an increasing amplitude.

You have now reached the end of this Programme and this brings you to the **Revision summary** and the **Can You?** checklist. Following that is the **Test exercise**. Work through this *at your own pace*. A set of **Further problems** provides additional valuable practice.

Revision summary 4

67

1 *Periodic functions*

$f(t) = f(t + nT) \qquad n = 1, 2, 3, \ldots \qquad$ Period $= T$.

2 *Laplace transform of a periodic function with period T*

$$L\{f(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} \cdot f(t)\,dt.$$

3 *Inverse transforms involving periodic functions*

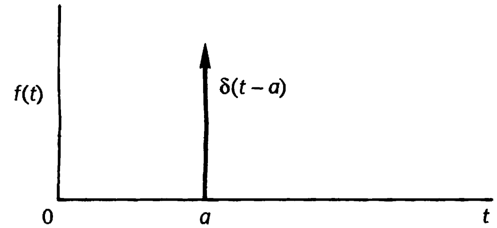
e.g. $L^{-1}\left\{\dfrac{1 + 2e^{-3s} - 3e^{-2s}}{s(1 - e^{-3s})}\right\}$

Expand $(1 - e^{-3s})^{-1}$ as a binomial series, like

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

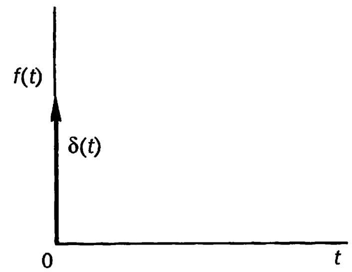
Multiply out and take inverse transforms of each term in turn. ▶

4 *Dirac delta function* or unit impulse function



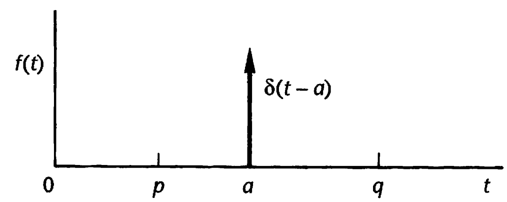
$$\begin{aligned} \delta(t-a) &= 0 \qquad t \neq a \\ &= \infty \qquad t = a. \end{aligned}$$

5 *Delta function at the origin*



$$a = 0 \qquad \therefore\ \begin{aligned} \delta(t) &= 0 \qquad t \neq 0 \\ &= \infty \qquad t = 0. \end{aligned}$$

6 *Area of pulse = 1*



$$\therefore \int_p^q \delta(t-a)\,\mathrm{d}t = 1$$

$$p < a < q$$

7 *Integration of the impulse function*

$$\int_p^q f(t)\cdot\delta(t-a)\,\mathrm{d}t = f(a) \qquad p < a < q$$

8 *Laplace transform of* $\delta(t-a)$

$L\{\delta(t-a)\} = e^{-as}$

$L\{\delta(t)\} = 1$ because $a = 0$

$L\{f(t)\cdot\delta(t-a)\} = f(a)\cdot e^{-as}$.

9 *Harmonic oscillators*

The equation of $af''(t) + bf(t) = 0$, $f(0) = \alpha$ and $f'(0) = \beta$, where a and b are of the same sign, represents a system undergoing simple harmonic motion and is referred to as an harmonic oscillator. The system oscillates with a frequency of $\sqrt{\dfrac{b}{a}}$ radians per unit of time and with period $\dfrac{2\pi}{\sqrt{b/a}} = 2\pi\sqrt{\dfrac{a}{b}}$ units of time. If a first derivative term is added to the left-hand side of the equation then, provided all three coefficients have the same sign, the system will undergo damped harmonic motion.

▶

10 *Forced harmonic motion*
Forced harmonic motion is achieved by the existence of a term on the right-hand side of the equation giving rise to transient and steady-state parts of the solution.

11 *Resonance*
Resonance is exhibited by a system undergoing periodic behaviour with a growing amplitude of vibration. Resonance occurs when a system, whose unforced behaviour is periodic, is forced with the same period.

Can You?

Checklist 4

68

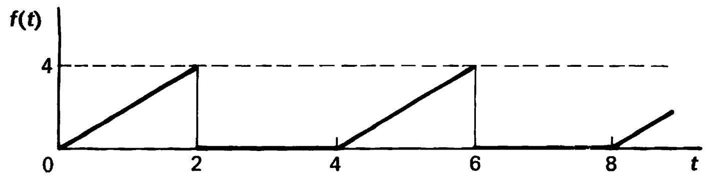
Check this list before and after you try the end of Programme test.

On a scale of 1 to 5, how confident are you that you can:	**Frames**
• Find the Laplace transforms of periodic functions?	1 to 14
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Obtain the inverse Laplace transforms of transforms of periodic functions?	15 to 28
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Describe and use the unit impulse to evaluate integrals?	29 to 34
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Obtain the Laplace transform of the unit impulse?	35 to 42
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Use the Laplace transform to solve differential equations involving the unit impulse?	43 to 51
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Solve the equation and describe the behaviour of an harmonic oscillator?	52 to 66
Yes ☐ ☐ ☐ ☐ ☐ *No*	

Test exercise 4

69

1 Determine the Laplace transform of the periodic function shown.



2 Evaluate

(a) $\int_0^4 e^{-3t} \cdot \delta(t-2)\,dt$

(b) $\int_0^\infty \sin 3t \cdot \delta(t-\pi)\,dt$

(c) $\int_1^3 (2t^2+3) \cdot \delta(t-2)\,dt.$

3 Determine (a) $L\{4 \cdot \delta(t-3)\}$, (b) $L\{e^{-3t} \cdot \delta(t-2)\}$.

4 Sketch the graph of $f(t) = 3 \cdot \delta(t) + 4 \cdot \delta(t-2) - 3 \cdot \delta(t-4)$ and determine its Laplace transform.

5 Solve the equation $\ddot{x} + 6\dot{x} + 10x = 7 \cdot \delta(t)$ given that, at $t = 0$, $x = -1$ and $\dot{x} = 0$.

6 The equation of motion of a system is
$\ddot{x} + 3\dot{x} + 2x = 3 \cdot \delta(t-4)$.
At $t = 0$, $x = 2$ and $\dot{x} = -4$. Determine an expression for the displacement x in terms of t.

7 Find the frequency, periodic time and solution for each of the following harmonic oscillators.

(a) $f''(t) + f(t) = 0$ given that $f(0) = 0$ and $f'(0) = 1$

(b) $6f''(t) + 2f'(t) + 9f(t) = 0$ given that $f(0) = 0$ and $f'(0) = 3$.

8 Find the transient and steady-state solutions of the forced harmonic oscillator
$f''(t) + 2f'(t) + 3f(t) = 4e^{5t}$ given that $f(0) = -2$ and $f'(0) = 6$.

Further problems 4

70

1 If $f(t) = a\sin t \quad 0 < t < \pi$, $= 0 \quad \pi < t < 2\pi$, $f(t + 2\pi) = f(t)$,

$$f(t) = \left.\begin{array}{ll} a\sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{array}\right\} \quad f(t+2\pi) = f(t),$$

prove that $L\{f(t)\} = \dfrac{a}{(s^2+1)(1-e^{-\pi s})}$.

2 If $f(t) = a\sin t \quad 0 < t < \pi \quad f(t+\pi) = f(t)$, determine $L\{f(t)\}$.

3 Find the Laplace transforms of the following periodic functions.

(a) $f(t) = t \qquad 0 < t < T \qquad f(t+T) = f(t)$

(b) $f(t) = e^t \qquad 0 < t < 2\pi \qquad f(t+2\pi) = f(t)$

(c) $f(t) = \left.\begin{array}{ll} t & 0 < t < 1 \\ 0 & 1 < t < 2 \end{array}\right\} \qquad f(t+2) = f(t)$

(d) $f(t) = \left.\begin{array}{ll} t^2 & 0 < t < 2 \\ 4 & 2 < t < 3 \end{array}\right\} \qquad f(t+3) = f(t)$

4 A mass M is attached to a spring of stiffness $\omega^2 M$ and is set in motion at $t = 0$ by an impulsive force P. The equation of motion is

$$M\ddot{x} + M\omega^2 x = P \cdot \delta(t).$$

Obtain an expression for x in terms of t.

5 An impulsive voltage E is applied at $t = 0$ to a series circuit containing inductance L and capacitance C. Initially, the current and charge are zero. The current i at time t is given by

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{q}{C} = E \cdot \delta(t)$$

where q is the instantaneous value of the charge on the capacitor. Since $i = \dfrac{\mathrm{d}q}{\mathrm{d}t}$, determine an expression for the current i in the circuit at time t.

6 A system has the equation of motion

$$\ddot{x} + 5\dot{x} + 6x = F(t)$$

where, at $t = 0$, $x = 0$ and $\dot{x} = 2$. If $F(t)$ is an impulse of 20 units applied at $t = 4$, determine an expression for x in terms of t.

7 Find the frequency, periodic time and solution for each of the following harmonic oscillators.

(a) $12f''(t) + f(t) = 0$ given that $f(0) = -1$ and $f'(0) = 2$

(b) $f''(t) + 12f(t) = 0$ given that $f(0) = 2$ and $f'(0) = -1$.

8 Solve for each of the following harmonic oscillators.

(a) $4{\cdot}6f''(t) + 2{\cdot}2f(t) = 0$ given that $f(0) = 1{\cdot}6$ and $f'(0) = -3{\cdot}1$

(b) $\sqrt{2}f''(t) + \sqrt{3}f(t) = 0$ given that $f(0) = 0$ and $f'(0) = \pi$.

9 Find the transient and steady-state solutions of the forced harmonic oscillator

$$4f''(t) + 3f'(t) + 2f(t) = e^t$$

given that $f(0) = 0$ and $f'(0) = 6$.